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On Townsend's rapid-distortion model of the turbulent-wind-wave problem

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Townsend's (1980) model of wind-to-wave energy transfer, which is based on a putative interpolation between an inner, viscoelastic approximation and an outer, rapid-distortion approximation and predicts an energy transfer that is substantially larger (by as much as a factor of three) than that predicted by Miles's (1957) quasi-laminar model, is revisited. It is shown that Townsend's interpolation effectively imposes a rapid-distortion approximation throughout the flow, rather than only in the outer domain, and that his asymptotic (far above the surface) solution implicitly omits one of the two admissible, linearly independent solutions of his perturbation equations. These flaws are repaired, and Townsend's dissipation function is modified to render the transport equation for the perturbation energy of the same form as those for the perturbation Reynolds stresses. The resulting wind-to-wave energy transfer is close to that predicted by Townsend's (1972) viscoelastic model and other models that incorporate the perturbation Reynolds stresses, but somewhat smaller than that predicted by the quasi-laminar model. We conclude that Townsend's (1980) predictions, although closer to observation than those of other models, rest on flawed analysis and numerical error.

1. Introduction

Townsend (1980) calculates the perturbation Reynolds stresses in a turbulent shear flow with the basic mean velocity $U(z) - c$ over the surface wave

$$z = a \cos kx \equiv h_0(x) \quad (ka \ll 1), \tag{1.1}$$

in a reference frame moving with the wave speed $c = (g/k)^{1/2}$, by interpolating between an inner, viscoelastic approximation and an outer, rapid-distortion approximation. He predicts a wind-to-wave energy transfer that is substantially larger (by as much as a factor of three) than that predicted by the quasi-laminar model (Miles 1957, 1959), in which the perturbation Reynolds stresses are neglected, and by other models that incorporate the turbulent Reynolds stresses (see Townsend 1972; Mastenbroek 1996; Mastenbroek *et al.* 1996; and the papers cited in § II.2.5 of Komen *et al.* 1994) but nevertheless predict energy transfers close to that of the quasi-laminar model. However, Townsend's (1980) predicted energy transfer is closer to observation, which, together with recent (Mastenbroek *et al.* 1996) experimental evidence of rapid distortion in turbulent flow over water waves (see Belcher & Hunt (1998) for a review of rapid distortion in the present context), suggests that his model be revisited. (All subsequent references to Townsend are to his 1980 paper unless explicitly noted otherwise.)

While it lies beyond the scope of this note to undertake a critical review of turbulent stress modelling in the wind-wave problem, for the sake of completeness we note that Belcher, Newley & Hunt (1993) introduce a novel proposal for determining the perturbation Reynolds stress based on the asymmetric pressure field induced by their mechanism of 'non-separated sheltering'. Their analytic solution, based upon a simple mixing-length closure hypothesis in the inner region, does yield larger values of energy-transfer coefficient β; indeed their figure 11 shows a coincidence of their prediction with that of Townsend. We simply note that the coincidence is just that: as our work here indicates, Townsend's value must be corrected considerably downward. There is, in any case, no apparent reason for the Townsend model, which lacks non-separated sheltering, to agree with Belcher *et al.* (1993), which has it.

We present here a modification of Townsend's model in which: (i) the Cartesian coordinates (x, z) are replaced by wave-following coordinates (ξ, η), which map the surface wave (1.1) on $\eta = 0$, thereby avoiding the projection of the boundary conditions in a region of strong shear; (ii) w_x, the streamwise derivative of the (Reynolds-averaged) vertical velocity, which he neglects, is restored in his approximation to the energy-transport equation; (iii) his dissipation length is generalized to allow for the departure of τ/e, the ratio of shear stress to turbulent intensity, from its equilibrium (undisturbed flow) value; (iv) his interpolation between the viscoelastic and rapid-distortion domains, which effectively imposes a rapid-distortion approximation throughout the flow rather than only in an outer domain, is replaced by a true interpolation; (v) his asymptotic ($kz \uparrow \infty$) solution, which implicitly omits one of the two admissible, linearly independent solutions to the fourth-order differential equation for the perturbation stream function, is corrected; (vi) his numerical integration is replaced by a more tightly controlled procedure. These modifications are, in our view, necessary to render the implementation of Townsend's model consistent with his verbal description of that model. They lead to a predicted energy transfer that is close to those of the other models cited above; however, most of the difference between Townsend's and the present results appears to reflect numerical error in Townsend's solution.

Townsend adopts the logarithmic profile

$$U(z) = (U_*/K)\log(z/z_0) \quad (z \gg z_0) \tag{1.2}$$

($U_* \equiv \tau_0^{1/2}$ is the friction velocity, $K \simeq 0.41$ is Kármán's constant, and z_0 is the effective roughness). This profile is singular at $z = 0$, but it may be replaced by

$$U(z) = (U_*/K)\log[1 + (z/z_0)], \tag{1.3}$$

which satisfies $U(0) = 0$. However, the energy transfer to the wave proves to be rather insensitive to the details of the flow in $z = O(z_0)$; see § 6.

The characteristic lengths for our problem are $1/k$, z_0, z_c (the elevation of the critical layer, at which $U = c$), and U_*^2/g, which are related by

$$\frac{z_c}{z_0} = \mathrm{e}^{Kc/U_*} - 1, \quad C_* \equiv \frac{gz_0}{U_*^2} = kz_0\left(\frac{c}{U_*}\right)^2, \tag{1.4a, b}$$

where C_* is Charnock's constant. ($\Omega \equiv gz_0/U_1^2 = K^2C_*$ in Miles (1959).) Following Komen *et al.* (1994), we choose c/U_* and C_* as the primary parameters in the subsequent development (Townsend chooses c/U_* and kz_0); a representative value of C_* is 1.44×10^{-2}, which implies $kz_0 \simeq 10^{-4}$ for $c/U_* = 12$. Two additional parameters (a_1 and b) appear through Townsend's Reynolds-stress closure (see § 3).

In § 2, we posit the Reynolds-averaged equations for the mean perturbation velocity and stresses and introduce a perturbation stream function in the wave-following coordinates. In § 3, as a first step towards a Reynolds-stress closure, we modify Townsend's energy-transport equation by restoring w_x (which he neglects) and neglecting the lateral transport of turbulent energy (which he models by a gradient-diffusion term).† More importantly, we modify his dissipation length (carried over from his 1972 paper) by allowing for the departure of τ/e from its equilibrium value a_1. This introduces an additional parameter μ, which we ultimately choose to render the transport equation for the perturbation energy of the same form as those for the perturbation Reynolds stresses.

In § 4, we consider the calculation of the wave-induced perturbation Reynolds stresses $\hat{\sigma} = -\langle u'^2 - w'^2 \rangle - \sigma_0$ and $\hat{\tau} = -\langle u'w' \rangle - \tau_0$ (all stresses herein are true stresses divided by density) in an inner domain, in which they satisfy a viscoelastic equation of the form

$$(\mathscr{D} + \lambda)[\hat{\sigma}, \hat{\tau}] = 2a_1[\sigma_0, \tau_0]\varepsilon, \tag{1.5}$$

where $\mathscr{D} \equiv (U - c)\partial_\xi$, $\lambda \equiv a_1/U'(\eta)$, σ_0 and τ_0 are the equilibrium values, and ε is a composite strain rate, and in an outer domain, in which Townsend posits a rapid-distortion approximation of the form

$$\mathscr{D}[\hat{\tau} - a_1\hat{e},\ \hat{\sigma} - b\hat{\tau}] = e_0[\varepsilon_A, \varepsilon_B], \tag{1.6}$$

where $\hat{e}$ is the wave-induced perturbation of the turbulent energy, $b = \sigma_0/\tau_0$, and ε_A and ε_B are linearly related to ε.

Townsend argues that 'the simplest way to interpolate between [the inner and outer approximations (1.5) and (1.6)]' is to replace $\mathscr{D}$ by $\mathscr{D} + \lambda$ in (1.6) to obtain

$$(\mathscr{D} + \lambda)[\hat{\tau} - a_1\hat{e},\ \hat{\sigma} - b\hat{\tau}] = e_0[\varepsilon_A, \varepsilon_B]. \tag{1.7}$$

We agree that this replacement renders the inner and outer transport equations of the same form, but, in our view, (1.7) remains an outer approximation, whereas Townsend imposes it throughout the entire flow.

We interpolate between (1.5) and (1.7) by multiplying $[\varepsilon_A, \varepsilon_B]$ by a function $I(\eta)$ that tends to 0/1 in the inner/outer limit $k\eta \to 0/\infty$. Moreover, we render the transport equation for $\hat{e}$ of the same form as (1.5) and (1.7) through a modification of Townsend's dissipation length.

In § 5, we assume monochromatic motion, for which the wave-induced perturbations are proportional to $\exp(ik\xi)$ and the partial differential equations of §§ 2–4 are reduced to a fourth-order, ordinary differential equation with two boundary conditions at $\eta = 0$ and two asymptotic matching conditions for $k\eta \uparrow \infty$. The ultimate goal of the calculation is the energy-transfer coefficient

$$\beta \equiv [(\rho_a/\rho_w)(U_*/c)^2]^{-1}(kc\mathscr{E})^{-1}(\partial\mathscr{E}/\partial t), \tag{1.8}$$

where ρ_a/ρ_w is the air–water density ratio and $\mathscr{E}$ is the (slowly changing) surface-wave energy.

In § 6, we present computations of β for a variety of parametric choices, including $I(\eta)$, $U(z)$, b, kz_0, and C_*. In sharp contrast to Townsend, all of our numerical results for the rapid-distortion model lie in the general range exhibited in previous computations for the quasi-laminar model (Miles 1959) and the closure models of

† The form of Townsend's gradient-diffusion term is questionable (Bradshaw, Ferriss & Atwell 1967), but both Townsend's (1972) results and those of Bradshaw *et al.* support its neglect in the present context.

Townsend (1972) and Mastenbroek (1996). We conclude that the most probable explanation of the discrepancy is numerical error in Townsend (1980).

Following some remarks on our integration scheme, we turn in the Appendix to a careful asymptotic expansion of far-field solutions to the governing fourth-order equation. Because of numerical considerations, the mixture of exponential and algebraic solutions makes for a delicate match, one that is often dispatched with, as in Townsend, by simple omission of the algebraic component. Such omission induces an artificial, though modest, oscillation in β as a function of c/U_*. We present numerical evidence that this match is accurately implemented in our computations.

2. Equations of motion

The Reynolds-averaged continuity and Euler equations for the mean perturbation velocity $[u, 0, w]$ in the Cartesian coordinates $[x, y, z]$ in the reference frame of the wave (1.1) are

$$u_x + w_z = 0, \tag{2.1}$$

$$(U - c)u_x + (\mathrm{d}U/\mathrm{d}z)w = -\pi_x + \sigma_x + \tau_z, \tag{2.2a}$$

and

$$(U - c)w_x = -\pi_z + \tau_x, \tag{2.2b}$$

where the subscripts x and z signify partial differentiation,

$$\pi \equiv \frac{p}{\rho} + \langle w'^2 \rangle, \quad \sigma \equiv -\langle u'^2 - w'^2 \rangle, \quad \tau \equiv -\langle u'w' \rangle, \tag{2.3a–c}$$

p is the mean perturbation pressure, ρ is the air density, u' and w' are the turbulent fluctuations of u and w, and $\langle\,\rangle$ implies a Reynolds average.

We mitigate the difficulties associated with the rapid variation of U in the neighbourhood of the interface by introducing the wave-following coordinates ξ and η through the transformation

$$x = \xi, \quad z = \eta + h(\xi, \eta), \tag{2.4a, b}$$

where $h(\xi, \eta)$ maps the interface $z = h_0(x)$ on $\eta = 0$ and vanishes far above ($k\eta \uparrow \infty$) the interface. A suitable choice proves to be (see § 6)

$$h(\xi, \eta) = h_0(\xi)\mathrm{e}^{-k\eta}. \tag{2.5a, b}$$

Introducing the stream function

$$\psi = \int_0^\eta [U(\eta) - c]\,\mathrm{d}\eta + [U(\eta) - c]h + \phi(\xi, \eta), \tag{2.6}$$

we cast the linearized mean velocity in the reference frame of the wave in the form

$$u = \psi_z \simeq (1 - h_\eta)\psi_\eta \simeq U - c + U'h + \phi_\eta \tag{2.7a}$$

and

$$w = -\psi_x \simeq -\psi_\xi + h_\xi \psi_\eta \simeq -\phi_\xi, \tag{2.7b}$$

where, here and subsequently, $U \equiv U(\eta)$ and $U' \equiv \mathrm{d}U/\mathrm{d}\eta$. Substituting (2.7) into (2.2) and eliminating π through cross-differentiation, we obtain

$$\partial_\xi[(U - c)(\phi_{\xi\xi} + \phi_{\eta\eta}) - U''\phi] = \sigma_{\xi\eta} + \tau_{\eta\eta} - \tau_{\xi\xi}. \tag{2.8}$$

Continuity of the interfacial velocity (we neglect viscosity and the wind-induced drift in the water) and evanescence of the wave-induced disturbance imply the boundary conditions

$$\phi = ch_0(\xi), \qquad \phi_\eta = (kc - U')h_0(\xi) \quad (\eta = 0) \tag{2.9a,b}$$

and

$$\phi, \sigma - \sigma_0, \tau - \tau_0 \to 0 \quad (k\eta \uparrow \infty). \tag{2.10}$$

3. Energy-transport equation

As a first step toward a Reynolds-stress closure, Townsend posits the transport equation for the turbulent energy $\langle q^2 \rangle /2$ in a form equivalent to

$$\mathscr{D}\langle \tfrac{1}{2}q^2 \rangle = D + G - E, \quad \mathscr{D} \equiv (U-c)\partial_\xi, \tag{3.1a,b}$$

where

$$D = \delta K U_* \partial_z [z \partial_z \langle \tfrac{1}{2} q^2 \rangle] \tag{3.2}$$

represents diffusion, which we henceforth neglect (Townsend chooses $\delta = 0.3$),

$$G = -\langle u'^2 \rangle u_x - \langle w'^2 \rangle w_z - \langle u'w' \rangle (u_z + w_x) \tag{3.3a}$$

$$\simeq \sigma_0 \phi_{\xi\eta} + \tau_0 [U'(\eta) + U''(\eta)h + \phi_{\eta\eta} - \phi_{\xi\xi}] + (\tau - \tau_0)U'(\eta) \tag{3.3b}$$

represents 'generation' (Launder, Reece & Rodi 1975), (3.3b) follows from (3.3a) through (2.1), (2.3b,c), (2.7a,b) and linearization, the subscript zero refers to the basic flow, and E is the dissipation rate. Townsend's approximation to G is equivalent to (3.3a) after neglecting w_x therein, but this neglect is unnecessary and inconsistent with his subsequent rapid-distortion approximation.

The dissipation rate is given by

$$E = (a_1 e)^{3/2}/L, \quad e \equiv \langle q^2 \rangle, \tag{3.4a,b}$$

where L is a dissipation length ($L \equiv a_1^{3/2} L_\varepsilon$ in Townsend's notation), and $a_1 \equiv \tau_0/e_0$ is the ratio of shear stress to turbulent intensity in the undisturbed flow. Townsend (1972) argues that L should be proportional to $z - h_0$ near the surface but 'more nearly proportional to height above the surface' for $kz = O(1)$ and posits

$$L = K(z - h_0 \mathrm{e}^{-kz}). \tag{3.5}$$

This is equivalent to $L = U_*/U'(\eta) \simeq K\eta$ for the logarithmic profile (with $\eta \gg z_0$) in the present, wave-following coordinates, but we allow for the departure of τ/e from its equilibrium value a_1 by choosing

$$L = \frac{U_*}{U'(\eta)} \left[1 + \mu \left(\frac{\tau - a_1 e}{\tau_0} \right) \right], \tag{3.6}$$

where μ is a constant (see §4.3). Combining (3.3b), (3.4a,b), (3.6) and

$$\hat{e} \equiv e - e_0, \quad \hat{\sigma} \equiv \sigma - \sigma_0, \quad \hat{\tau} \equiv \tau - \tau_0 \tag{3.7a–c}$$

and neglecting the diffusion term D in (3.1a), we obtain

$$(\mathscr{D} + \lambda) a_1 \hat{e} = 2a_1 \tau_0 \varepsilon + 2\lambda(1+\mu)(\hat{\tau} - a_1 \hat{e}), \tag{3.8}$$

where

$$\varepsilon \equiv U''h + \phi_{\eta\eta} - \phi_{\xi\xi} + b\phi_{\xi\eta}, \tag{3.9}$$

$$a_1 \equiv \tau_0/e_0, \quad b \equiv \sigma_0/\tau_0, \quad \lambda \equiv a_1 U'. \qquad (3.10a–c)$$

Townsend chooses (implicitly) $a_1 = 1/6$. We choose $a_1 = K^2$, which differs insignificantly from 1/6 (0.168 vs. 0.167). He does not specify his value of b, but we surmise from his reference to Launder *et al.* (1975) that he used their value, $b = -1.3$, which also is adopted by Mastenbroek (1996).

4. Reynolds-stress closure

We consider separately the calculation of the perturbation Reynolds stresses in an inner (viscoelastic) domain in which $k|U - c| \ll a_1 U'$ and an outer (rapid distortion) domain in which $k|U - c| \gg a_1 U'$.

4.1. *Viscoelastic domain*

Following Townsend, we assume that the perturbation stresses in the inner domain tend to their equilibrium values:

$$\hat{\tau} = a_1 \hat{e}, \quad \hat{\sigma} = b\hat{\tau}. \qquad (4.1a,b)$$

Eliminating $\hat{e}$ between (3.8) and (4.1a), we obtain the viscoelastic equation

$$(\mathscr{D} + \lambda)\hat{\tau} = 2a_1\tau_0\varepsilon, \qquad (4.2)$$

in which the effective strain rate ε is given by (3.9). The effective viscosity is $2a_1\tau_0/(\mathscr{D}+\lambda)$, which tends to the mixing-length limit $2KU_*\eta$ for $k\eta \downarrow 0$.

4.2. *Rapid-distortion domain*

In the outer domain (in which 'the time scales of the undisturbed flow are long compared with the time scale of the wave perturbation following the mean flow'), Townsend posits rapid-distortion approximations equivalent to

$$\mathscr{D}\left(\frac{\hat{\tau} - a_1\hat{e}}{e_0}\right) \sim A_1(U''h + \phi_{\eta\eta}) - A_2\phi_{\xi\xi} + A_3\phi_{\xi\eta} \equiv \varepsilon_A \qquad (4.3a)$$

and

$$\mathscr{D}\left(\frac{\hat{\sigma} - b\hat{\tau}}{e_0}\right) \sim B_1(U''h + \phi_{\eta\eta}) - B_2\phi_{\xi\xi} + B_3\phi_{\xi\eta} \equiv \varepsilon_B, \qquad (4.3b)$$

where A_n and B_n (which are plotted vs. $U'/\lambda = 1/a_1$, Townsend's 'total shear', in his figure 5) are 'the incremental rates of change for suddenly imposed additional distortions.' (See Belcher & Hunt (1998) for a discussion of rapid-distortion scaling.)

Townsend argues that 'the simplest way to interpolate between [the inner and outer approximations]' is to replace $\mathscr{D}$ by $\mathscr{D} + \lambda$ in (4.3) to obtain

$$(\mathscr{D} + \lambda)(\hat{\tau} - a_1\hat{e}) = e_0\varepsilon_A, \quad (\mathscr{D} + \lambda)(\hat{\sigma} - b\hat{\tau}) = e_0\varepsilon_B. \qquad (4.4a,b)$$

We agree that this replacement is necessary for the matching of the inner and outer approximations, but it is sufficient only if $A_n = B_n = 0$, and we therefore regard (4.4) as an outer approximation.

4.3. *Interpolation*

We interpolate between (4.2) and (4.4) by multiplying ε_A and ε_B in (4.3a,b) by an interpolation function $I(\eta)$ that tends to 0/1 in the inner/outer limit $k\eta \to 0/\infty$. Moreover, we reduce (3.8) to the same form as (4.2) and (4.4),

$$(\mathscr{D} + \lambda)a_1\hat{e} = 2a_1\tau_0\varepsilon, \qquad (4.5)$$

by choosing $\mu = -1$. Eliminating $\hat{e}$ through (4.5) then yields

$$(\mathscr{D} + \lambda)\hat{\tau} = 2a_1\tau_0[(1 + \hat{A}_1)(U''h + \phi_{\eta\eta}) - (1 + \hat{A}_2)\phi_{\xi\xi} + (b + \hat{A}_3)\phi_{\xi\eta}], \tag{4.6a}$$

and

$$(\mathscr{D} + \lambda)(\hat{\sigma} - b\hat{\tau}) = 2a_1\tau_0[\hat{B}_1(U''h + \phi_{\eta\eta}) - \hat{B}_2\phi_{\xi\xi} + \hat{B}_3\phi_{\xi\eta}], \tag{4.6b}$$

where

$$\hat{A}_n \equiv \frac{A_n}{2a_1^2}I, \quad \hat{B}_n \equiv \frac{B_n}{2a_1^2}I. \tag{4.7a, b}$$

The viscoelastic equation (4.2) is recovered for $I = 0$. Townsend's formulation, except for the present use of wave-following coordinates, the restoration of w_x in G(3.3), and the replacement of (3.5) by (3.6), is recovered for $I = 1$.

5. Monochromatic motion

The wave-induced perturbations admit the representation

$$[h, \phi, \pi, \hat{\sigma}, \hat{\tau}] = \mathrm{Re}\{[H, \Phi, P, S, T]\mathrm{e}^{\mathrm{i}k\xi}\}, \tag{5.1}$$

where $H \ldots$ are complex functions of η. Transforming (2.8), (2.2a), (4.6a, b) and (2.9), we obtain

$$\mathscr{L}\Phi \equiv \mathrm{i}k[(U - c)(\Phi'' - k^2\Phi) - U''\Phi] = \mathrm{i}kS' + T'' + k^2T, \tag{5.2}$$

$$P = S + (\mathrm{i}k)^{-1}T' + U'\Phi - (U - c)\Phi', \tag{5.3}$$

$$T = \left(\frac{2a_1\tau_0}{\mathscr{D} + \lambda}\right)[(1 + \hat{A}_1)(U''H + \Phi'') + k^2(1 + \hat{A}_2)\Phi + \mathrm{i}k(b + \hat{A}_3)\Phi'] \tag{5.4a}$$

and

$$S = bT + \left(\frac{2a_1\tau_0}{\mathscr{D} + \lambda}\right)[\hat{B}_1(U''H + \Phi'') + k^2\hat{B}_2\Phi + \mathrm{i}k\hat{B}_3\Phi'], \tag{5.4b}$$

where $\hat{A}_n$ and $\hat{B}_n$ are given by (4.6), $\mathscr{D} = \mathrm{i}k(U - c)$, and

$$\Phi = ac, \qquad \Phi' = a(kc - U') \quad (\eta = 0). \tag{5.5a, b}$$

We remark that (5.2) reduces to Rayleigh's equation if the perturbation Reynolds stresses are neglected.

We seek the solution of (5.2)–(5.4), subject to (5.5) and null conditions for $k\eta \uparrow \infty$, and the corresponding interfacial impedance (cf. Miles (1957), in which α and β are referred to kaU_1^2, $U_1 \equiv U_*/K$)

$$\alpha + \mathrm{i}\beta \equiv (kaU_*^2)^{-1}(P + \mathrm{i}T)_0 \tag{5.6a}$$

$$= (kaU_*^2)^{-1}[(b + \mathrm{i})T + (\mathrm{i}k)^{-1}T']_0 + (c/U_*)^2, \tag{5.6b}$$

where (5.6b) follows from (5.6a) through (5.3), (5.5) and the inner limit $S = bT$.

6. Numerical results

First we present results taken from both Townsend (1980) and Townsend (1972) in comparison with our own computation of β as given in (5.6). We then consider variations on the computation of β to establish the degree of sensitivity to the choice

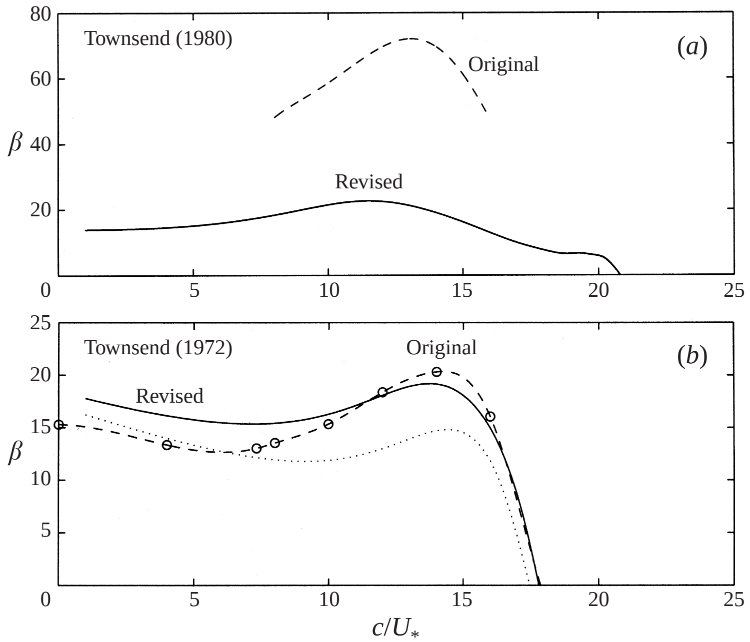

FIGURE 1. A plot of the normalized drag determined from (5.6*a*) as the sum of the imaginary component of the perturbation pressure and the real component of the perturbation shear stress. (*a*) Townsend's (1980) rapid-distortion model: the dashed line is the sum of curves in Townsend (1980) figures 6(*a*) and 6(*b*) corresponding to $KU(k^{-1})/\tau_0^{1/2} = 8$ in his notation, and the solid line indicates our revision. (*b*) Townsend's (1972) viscoelastic model: the circles are directly from Townsend's table 2 on summing the entries labelled P_i and τ_r, the dashed line is our smoothed fit, the solid line is our revision, and the dotted line is the present viscoelastic approximation ($I = 0$).

of interpolation in Reynolds stress parameterization from the outer to the inner region, the prescribed value of b, and to the form adopted for the wave-following coordinate. For reference, we show several computations others have pursued using different closure schemes. The only major change we observe is the initial correction to Townsend (1980).

The most striking aspect of figure 1(*a*) is the large offset of the curve marked 'original' (after Townsend), which has a peak value at least three times any other maximum exhibited here. We considered a variety of effects that might contribute to the dramatic reduction reported here ('revised') and summarize those findings presently.

An apt comparison for both curves is Townsend's (1972) seminal extension of the quasi-laminar model (Miles 1957) to incorporate the perturbation Reynolds stresses. The open circles in figure 1(*b*) are taken directly from Townsend's table 2. As the 1972 model is a special case of (5.2), with $I = 0$, $b = 0$ and w_x neglected, we have repeated that computation here. As can be seen in figure 1(*b*), the results agree tolerably well with Townsend (1972).

Following Komen *et al.* (1994, §II.2.3), we use the condition of constant C_* to fix the velocity profile in the three solutions of (5.2) illustrated in figure 2(*a*). These span a range of interpolants, as I proves the most sensitive factor in the solution of (5.2)–(5.4). As noted earlier, the choice $I = 1$ amounts to imposing the rapid-distortion

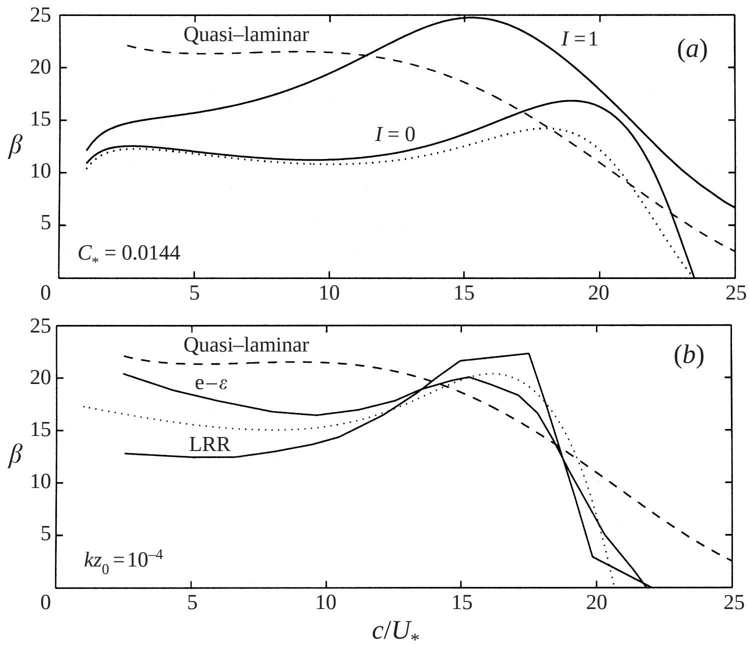

FIGURE 2. (a) The solution of (5.2)–(5.4) for the logarithmic velocity profile with Charnock's $C_* = 0.0144$ and three different choices of I: $I = 1$, the rapid-distortion approximation carried all the way to the surface; $I = 0$, the present viscoelastic limit; and $I = 1 - \exp(-\eta)$, an interpolation between these limits. The quasi-laminar result (– – –) is shown for comparison. (b) Two closure models after Mastenbroek (1996): the e–ε model and the Launder *et al.* model. The dotted line is the prediction derived from Townsend's (1972) model, adjusted to kz_0 fixed at 10^{-4}, the value used in Mastenbroek.

approximation all the way to the surface. This yields the curve designated by '$I = 1$' in figure 2(a) (also the curve labelled 'revised' in figure 1a).

The viscoelastic result in figures 2(a) and 1(b) (the dotted line), obtained with $I = 0$, is more like the quasi-laminar result in that the response is less peaked at larger c, but the magnitude of β is considerably less at small c. It seems desirable that $I(\eta)$ be a function that attains the viscoelastic limit in the inner region and the rapid distortion limit in the outer region. A purely heuristic choice is $I = 1 - \exp(-\nu k\eta)$. The β that emerges, however, does *not* lie between the nominally limiting curves with $I = 0$ and $I = 1$, but instead lies *below* $I = 0$ in the case shown (dotted line, for which $\nu = 1$). One obvious consideration is that the derivatives of this interpolant are non-zero, indeed large near the origin, so the limit of $\nu \to \infty$ should not be expected to, and does not, tend smoothly to the $I = 1$ result. Further comments on the interpolation appear in the Appendix.

Finally, in figure 2(b), we reproduce two curves after the results from two closure models previously illustrated in Mastenbroek (1996). The quasi-laminar result provides a useful point of comparison, and lastly we have introduced a dotted line to indicate the prediction of the Townsend (1972) model, adjusted for $kz_0 = 10^{-4}$.

Setting aside the purely numerical issues, to which we return later in this section, one seeks to discover, beyond the interpolation noted above, which other parameters of the many in the rapid-distortion model, the basic velocity profile, and the wave-following coordinates significantly affect the interfacial impedance.

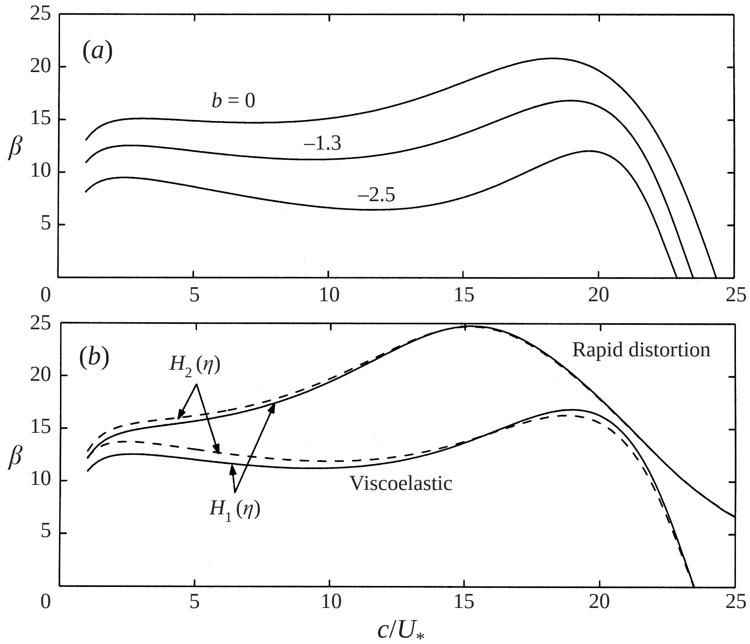

FIGURE 3. (a) Using the viscoelastic limit ($I = 0$) from figure 2(a), the parameter b is varied as a test of sensitivity. Note that $b = -1.3$ is a generally accepted value. (b) Two parameterizations of the wave-following coordinate: the solid lines (curves taken from figure 2b) use the simple exponential form $H_1(\eta)$, the dashed lines show the slight changes with the more elaborately constructed $H_2(\eta)$.

A subtle feature of (5.2)–(5.4) is the delicate relation of A_1 to K. As outlined in the Appendix, there is an algebraically decaying, far-field solution with a decay rate of $-K^2(A_1 + 2K^4)^{-1/2}/2$. Thus if $A_1 < -2K^4 \simeq -0.0565$, we can anticipate a large qualitative change in β. (This was confirmed with a numerical experiment with A_1 set equal to -0.055.)

We digitally scanned Townsend's figure 5 to extract values for the A_n and B_n. After conversion to proper units, we fixed these at $A_n = [-0.0315, 0.1693, 0.2274]$ and $B_n = [0.2204, 0.7918, 1.7388]$. His constant A_1 is consistently negative for all a_1 (reciprocal of the total shear) less than about 1/2, but remains comfortably shy of the critical value.

As noted above, figure 2(a) differs from figures 1(a,b) and 2(b) in the assumed base profile: constant C_* and constant kz_0 respectively. The two results for $I = 1$, the rapid-distortion model, cannot be brought into precise coincidence, but in rough terms, the Charnock result is shifted rightward by about $\Delta(c/U_*) = 4$ and upward by $\Delta\beta = 2$ relative to the result for constant $kz_0 = \exp(-8)$; the change is modest.

So too for variation of $\beta(c)$ with changes in b: computations of the viscoelastic curve of 2(a) with $b = 0$ and $b = -2.5$ are generally parallel, as shown in figure 3(a). Two other features have – as one might anticipate – only a minor effect: the form of the wave-following coordinate induced by $h(\xi, \eta)$, and the precise form of the velocity profile in the inner region. The first of these is indicated in figure 3(b). The notation is as follows:

$$H_1(\eta) = a\mathrm{e}^{-k\eta} \tag{6.1a}$$

and

$$H_2(\eta) = a(1-\boldsymbol{k})^{-2}\{\mathrm{e}^{-k\eta} - [2\boldsymbol{k} - \boldsymbol{k}^2(1-\boldsymbol{k})k\eta]\mathrm{e}^{-\eta/z_0}\}, \qquad \boldsymbol{k} \equiv kz_0. \qquad (6.1b, c)$$

The second form, which in addition to $H(0) = 1$, satisfies $H' = H'' = 0 \quad (\eta = 0)$, is intended to mitigate the effects of the large derivatives U''' and U'''' in the calculation of S' and T'' near the surface. In practice, this has not proved a difficulty. While the particular choice of wave-following coordinates has some effect on the computed β, it is clear from figure 3(b) that the effect is slight, as one would expect.

We have also experimented with a second velocity profile, given by (cf. Rotta 1950)

$$KU/U_* = \log[\zeta + (\zeta^2+1)^{1/2}] - \zeta[\zeta + (\zeta^2+1)^{1/2}]^{-1}, \qquad (6.2)$$

where $\zeta = ez/(2z_0)$. This form is a constant-stress interpolation from the surface condition $U = 0$ to a logarithmic profile for large z. The results for β are perturbed a maximum of two percent at $c = 1$, and the difference rapidly decreases to a fraction of a percent for increasing c.

We defer a discussion of the proper implementation of boundary conditions to the Appendix, where an asymptotic analysis is set out. For the purpose of the present discussion, the essential point is that even with a crude choice of the boundary condition, including even the imposition of a solid wall at large η, the value of β is not greatly perturbed.

While to bring our formulation into exact conformity with Townsend would require that we omit the wave-following coordinates, on the evidence of figure 3(b) this is not a critical factor in comparing results. Similarly, reversion to Townsend's form for L in (3.5) is unlikely to effect a substantial increase in β. While there are certain mathematical concerns that can complicate the solution for large c if μ is set to zero in (3.6) (or indeed any value other than -1), over the range $10 \leqslant c \leqslant 15$ computations are straightfoward and show at best moderate changes in β.

Finally, given the similarity of the viscoelastic model ($I = 0$) and the revised computation for Townsend (1972), it is implausible that omission of w_x from G in (3.3) – the final discrepancy between (5.2) and Townsend – would do much to close the gap.

In summary, we can find nothing in the formulation of (5.2)–(5.4), or immediately adjacent models, that could give rise to the anomalously large values of β reported in Townsend. The discussion in that paper is short on details of the numerical solution, but, by process of elimination of other sources for that variance, we are left with the tentative conclusion that the numerical values reported there are in error.

6.1. *Comments on numerical issues*

The boundary layers characteristic of the solution of (5.2)–(5.4) plainly make it desirable to use an adaptive step size integrator for the numerical solution. The sensitivity of β to roundoff error encourages one to use extrapolation for its efficient approach to high accuracy. Indeed, we find that Richardson-polynomial extrapolation is a factor of six faster than a fourth-order Runge–Kutta algorithm when both are run with a local relative error tolerance fixed at 10^{-12}. Speed aside, for a fixed tolerance, we find that the values of β produced are completely stable to switching among any of the three independent programs we tested. Moreover, further refinement of the tolerance to 10^{-14} makes inconsequential changes in computed values of β.

While we have preferred, for reasons outlined in the Appendix, to compute solutions integrating from large η to the surface, we have in a few test cases confirmed that substantially the same values of β can be found by integrating in the other direction.

Errors can be introduced at many stages in the transition from equation to code. We have tried to minimize these by extensive use of Maple to autogenerate the Fortran source code whenever possible. Although it is hard to find precise test cases for (5.2), the general conformity of the values computed here for figure 2(*b*) with earlier results in Mastenbroek (1996) is reassuring. The most rigorous consistency test of independently determined quantities is the excellent agreement of the computed and predicted asymptotic behaviour (figure 4*b* in the Appendix).

7. Conclusions

We conclude that the wind-to-wave energy transfer predicted by Townsend's rapid-distortion model (after incorporating the present modifications) differs from that predicted by the quasi-laminar model by an amount that is comparable with the differences among other models that incorporate the wave-induced Reynolds stresses and is smaller than the difference between any of these predictions and observation. This is perhaps surprising, since the critical-layer singularity that plays a crucial role in the quasi-laminar model is eliminated in these other models. Thus, Mastenbroek (1996) remarks that 'When the effect of turbulence was taken into account ... all calculations showed that the wave-induced turbulence is responsible for the phase shift in the pressure that is required for an energy flux to waves.' And Townsend (1972) opines that 'It is surprising that the differences [between the predictions of the quasi-laminar and viscoelastic models] are not larger since the critical layer ... is of central importance in [the quasi-laminar model] while it is merely an unimportant part of an equilibrium layer if turbulent stresses are included through the turbulent energy equation.' But the essential question here is whether the phase jump across the critical-layer singularity in the quasi-laminar *model* provides a valid approximation to the energy transfer from mean to disturbed flow. The situation is reminiscent of that for stability of a viscous shear flow, in which context Lin (1955, p. 119 ff) remarks that 'We have thus an adequate formulation ... in terms of a differential equation of the second order [Rayleigh's equation]. Such a theory should apply if the fluid has no viscosity to begin with. Whether it gives a proper limiting theory for the viscous problem has to be examined very carefully.' This last question is essentially resolved in the present context by a calculation of the Reynolds stress $-\rho\overline{uw}$ (temporal mean value in the quasi-laminar flow), the discontinuity of which across the critical layer gives the wind-to-wave energy transfer; cf. Taylor (1915).

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Appendix

Using the standard exponential substitution of the Liouville–Green method for the asymptotic solution of (5.2), it is straightforward, if tedious, to uncover an expansion of the phase function

$$\Phi \sim \hat{\eta}^{\gamma_1}(\log(\hat{\eta}/z_0))^{\gamma_2}\exp(\gamma_3/\log(\hat{\eta}/z_0))\ldots\exp(\mathrm{i}[\alpha_1\hat{\eta}\log(\eta/z_0)+\alpha_2\hat{\eta} \\ +\alpha_3\mathrm{Ei}(1,-\log(\hat{\eta}/z_0))+\alpha_4\hat{\eta}/\log(\hat{\eta}/z_0)+\cdots]), \qquad (\mathrm{A}\,1)$$

where the α_k and γ_k are real, and we define $\hat{\eta} \equiv k\eta$. Formally there is an infinite number of terms with coefficients α_k that precede the determination of the γ_k. However, as only the first of these, α_1, enters into the formula for γ_1, the first two into γ_2 (although the α contributions happen to cancel), the first four into γ_3, and so on, the expansions can be done in parallel.

As the governing equation is fourth order, we find α_1 by solving for the roots of the fourth-degree polynomial

$$\alpha_1^2[(2K^4 + A_1)\alpha_1^2 + 1] = 0. \tag{A 2}$$

Two roots are zero; the other two constitute an imaginary pair. (This is not automatic as Townsend's value of A_1 is negative.) The double zero leads to a particularly simple result at next order: $\alpha_2^2 - 1 = 0$, so that one pair of solutions is approximately $\exp(\pm\eta)$. The pure imaginary root pair exhibits weak algebraic growth (or decay), as reflected in γ_1. A suitable boundary condition for (5.2) is that the numerically integrated solution match smoothly onto a linear combination of the decaying exponential and the decaying algebraic solution. For that purpose, we employed the following expressions:

$$\alpha_1 = -(A_1 + 2K^4)^{-1/2}, \tag{A 3a}$$

$$\alpha_2 = (1 + Kc)(A_1 + 2K^4)^{-1/2} - \frac{(A_3 + B_1)}{2(2K^4 + A_1)} - \frac{b(4K^4 + A_1)}{2(2K^4 + A_1)}, \tag{A 3b}$$

$$\gamma_1 = -\tfrac{1}{2}K^2(A_1 + 2K^4)^{-1/2}, \quad \gamma_2 = -3/2 \tag{A 4a, b}$$

(after selecting $a_1 = K^2$) and a similar, though somewhat simpler, set corresponding to the real exponential pair. Also, we did in each case carry through the analysis of $\alpha_{3,4}$ and γ_3 but omit the expressions here because of their length. (Although the Ei function arises in (A 1), actually it is not needed. We choose our initial condition such that $\Phi = 1$, hence only the derivative of the Liouville–Green phase function, $S'(\eta)$, is required in computation.)

The existence of a *growing* exponential solution means that numerical solutions can only be carried a modest distance when integrating outward from the surface; hence a preferable strategy is to integrate from large η *to* the surface. The coefficients in the asymptotic expansion above can then be used to compute starting values of the functions and their derivatives provided η is sufficiently large. However, η cannot be too large or else the desired exponential ($\exp(-\eta)$) will swamp the algebraic solution. We have adopted $\eta = 18$ as a reasonable compromise to resolve a satisfactory match.

Townsend (1980) evidently (cf. Townsend 1972) matched with a condition of pure exponential decay, integrating outward from the surface. Whenever the match chosen has a crude or non-existent relation to the correct far-field phase function, the resulting β inevitably has coarse oscillations as c (or any other parameter) is varied. Our trials with matches similar to Townsend's have hence proved less than satisfactory, although the results so obtained do not diverge sufficiently far from a more refined computation to make a qualitative difference in conclusions about β. The likely corollary is that the large discrepancy between our results for β and Townsend's depends only marginally upon a differing approach to the implementation of boundary conditions.

Figure 4 shows a typical computed solution with the logarithm of the amplitude exhibiting a smooth transition from the initial exponential phase to the far slower algebraic behaviour at large η. Notice that the solution right up to $\eta = 18$ remains quite smooth. If fewer terms are kept in (A 1), one observes a significant adjustment region in the vicinity of $\eta = 18$ that is attributable to an admixture of the other two homogeneous solutions. In the expanded view in figure 4(b) a solid line is superposed

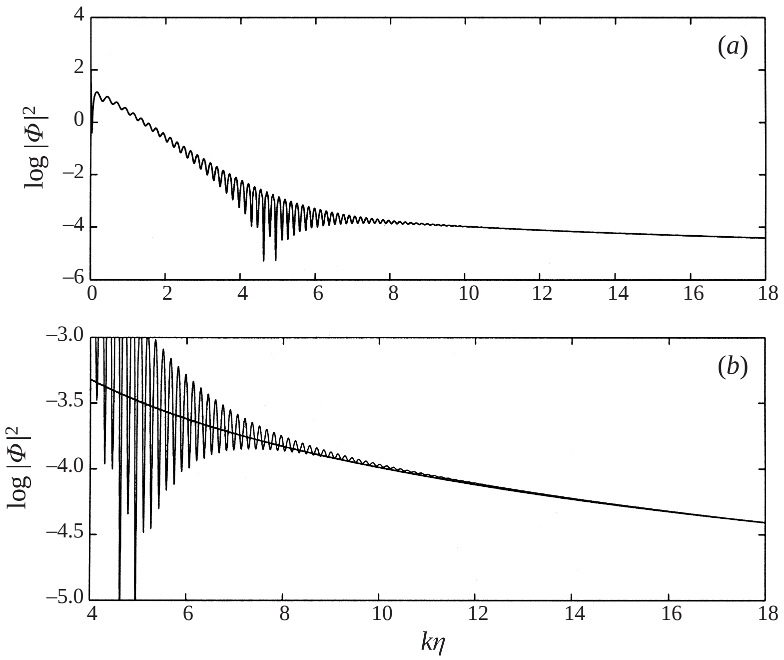

FIGURE 4. Match to two decaying fields at $\eta = 18$ for $c = 15$: (a) shows initial exponential decay followed by algebraic tail; (b) shows closeup of the algebraic tail, with a superposed line that is computed according to the asymptotic expansion in (A 1).

to indicate the $k\eta$ dependence of $|\phi|^2$ predicted by (A 1) under the assumption that only the algebraic mode is present. (The amplitude contribution of the exponential mode is simply too small to register on the plot when $k\eta$ is near 18.) A computation of the phase of ϕ (not shown) indicates comparable agreement: a drift of about $\pi/10$ occurs over six periods.

Within the scope of this model, the most significant determinant of β is the choice of the interpolant, I. Although we do not see a compelling reason, only convenience and aesthetics, to adopt the choice of $\mu = -1$, we note that for any other choice the computation of β can be problematic. The difficulty arises from the resultant denominator, $\mathscr{D} + \lambda$, which, at the critical layer, though not vanishing, nonetheless undergoes rapid change. While at small values of c, for which $k\eta \ll 1$, this is not a particular problem, when it occurs at $k\eta$ of order one, as is the case for large c, the value of β undergoes extreme variation in magnitude and implausibly rapid oscillations in sign.† The only way to avert this is to invoke some smoothing argument to adjust the interpolant in the vicinity of the critical layer, but there is no internal guidance from rapid-distortion theory *per se* as to how this might be done. Indeed we must in this respect differ from Townsend's earlier quoted view that the critical layer is 'merely an unimportant part of an equilibrium layer if turbulent stresses are included through the turbulent energy equation'. The matter does not disappear entirely; it leaves a potentially awkward vestige, and for this reason we have preferred to stay with $\mu = -1$ in § 4.3.

† Even with the choice of $\mu = -1$ we find β can oscillate in sign for c larger than about 25. It is questionable, however, to rely upon a pure logarithmic velocity profile so far from the surface, and we have not explored the balance of terms that generates the oscillation.

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